Synchronizability of a Type of Stochastic Oscillator Networks

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Abstract

In the present paper we investigate the conditions for synchronizability of a network of stochastic oscillators. In particular we consider stochastic extension of a model of weakly connected oscillatory neural networks. We employ methods from the stability theory of stochastic differential equations in order to find criteria for stability of synchronized states in the presence of random perturbations. Necessary and sufficient conditions for existance of stable synchrony are derived in the limit case of small noise.

Keywords: synchronizability, oscillator networks, stochastic oscillators, stochastic differential equations, stochastic stability.

1 Introduction

The mechanisms behind the signal processing in ensembles of neurons are still far from being fully understood. Among the variety of phenomena occurring in the dynamics of such complex systems the phenomenon of synchronization seems to play here an important role. The ability of a system to reach a synchronized state possibly affects its signal processing characteristics. Oscillatory and synchronous behaviour is found in a wide range of physical systems, from laser to brain dynamics (see e.g.[21, 11] and references therein). One of the possible approaches to study the oscillatory behaviour of neurons is the analysis of networks of limit cycle oscillators.
Among the multitude of models of neural oscillators the Hodgkin-Huxley model or the Morris-Lecar model and the the FitzHugh-Nagumo models shall be mentioned as they have received much attention as one of the simplest two dimensional models with nontrivial behaviour (cf. [6, 19 and 7] for an overview). Still, from in vivo observations it is known that single neurons usually cannot be considered as fully deterministic phase oscillators. The appearance of highly irregular and hardly predictable activity gives some reason to assume an influence of noise on the neural dynamics. The possible sources of noise are intrinsic (like the thermal noise) as well as extrinsic (like those caused by synaptic transmission interrupts).

In spite of this ubiquity of noise in neural systems there are only few papers dealing (employing rigorous analytical methods) with stochastic variants of the well known models (in this context the book of Pikovsky et al., [18], shall be named, where mainly the physical approach is discussed). Among those, the determination of firing times for the stochastic FitzHugh-Nagumo model by Tuckwell et al. [24] shall be mentioned here and the analysis of noise induced synchronization ([4 and the references therein). For other analytical approaches see also [26, 3, 23, 13].
In the present work we consider a stochastic version of a particular model system of coupled oscillators. In the deterministic case each oscillator exhibits stable limit cycles. We address the question of the robustness of these states in the presence of random perturbations. we are in particular interested in conditions of synchronizability of the system and the influence of the connection topology in this context. With this approach we intend to contribute also to the applications concerning the modelling of epileptic seizures, where stochastic synchronization and desynchronization is believed to play an important role (see, e.g. [15]).
The paper is organized as follows. The oscillator model as the object of our investigations is shortly introduced in section 2. In section 3 we review the stochastic dynamics of a single neuron. Further, the analysis of a network of weakly connected neurons with N neurons and a given network structure is given in section 4. In section 4.3 we give an example of a desynchronization of the oscillators. In this case the phase difference between the oscillators takes values given by a probability density centered around zero. Finally, in section 5 we summarize our results.

2 Description of the model

The time evolution of a single oscillator of this type may be represented by the following complex valued differential equation (cf. [10])

$$\dot{z} = (\alpha + i\omega)z - z|z|^2, \qquad \alpha, \omega \in \mathbb{R}, \tag{1}$$

where α plays the role of a bifurcation parameter (with a bifurcation point at $\alpha = 0$) or an abstract intput energy, and ω is a natural frequency parameter. Stable oscillations occur for $\alpha > 0$.
Transformed to polar coordinates, $z = re^{i\phi}$, $r \geq 0, \phi \in [0, 2\pi)$ one obtains a two dimensional equation of the form

$$\begin{aligned} \dot{r} &= \alpha r - r^3 \\ \dot{\phi} &= \omega \end{aligned} \tag{2}$$

The corresponding model for a system of n oscillators from (1) is then given by

$$\dot{z_k} = (\alpha_k + i\omega_k)z - z_k|z_k|^2 + \sum_{k\neq l}^{n} c_{kl}z_l, \qquad k,l = 1..n. \tag{3}$$

Remark 2.1 *The model given by*

$$\dot{z_k} = (\alpha_k + i\omega_k)z + (\sigma_k + i\gamma_k)z_k|z_k|^2 + \sum_{k\neq l}^{n} c_{kl}z_l, \qquad k,l = 1..n. \tag{4}$$

with $\sigma_k, \omega_k \in \mathbb{R}$ is sometimes called 'canonical' for systems of the form $\dot{x}_i = f_i(x_i,\lambda) + \epsilon g_i(x,\lambda)$ being near an Andronov-Hopf bifuracation and where $\lambda = \lambda(\epsilon)$ is the (real valued) bifurcation parameter with $\lim_{\epsilon\to 0} \frac{1}{\epsilon}\lambda(\epsilon) = \lambda_0$ ([10]). □

3 Single oscillator

Consider the (Itô-) generalization of equation (1) by (a multiplicative Gaussian white noise) perturbation of the bifurcation parameter $\alpha r dt \mapsto \alpha r dt + \eta r dB_t$, $\eta \geq 0$ (which is in fact only one of several possible ways to introduce the noise into (1), for another example of a treatment of an additive noise perturbation see e.g. [1]). With this one obtains the following equations from (2)

$$dr = (\alpha r - r^3)dt + \eta r dB_t, \quad r(0) = r_0 > 0 \tag{5a}$$

$$d\phi = \omega_0 dt, \qquad \phi(0) \equiv \phi_0 \in [0, 2\pi) \tag{5b}$$

Note that in general there is no experimental evidence that noise sources in biological systems always produce white noise. Here, the white noise perturbation of α shall be understood as an approximation of a general noisy input.
We also note that the equations are uncoupled, and of course (5b) is solved by $\phi(t) = \omega_0 t + \phi_0$. Equation (5a) is on the other hand equivalent to

$$du = (\alpha - e^{2u} - \frac{1}{2}\eta^2)dt + \eta dB_t, \qquad u = \ln r, r > 0, u(0) = \ln r_0 \tag{6}$$

We note that in this representation the noise is additive.

Lemma 3.1 *The process $r(t)$ as solution of (5) with $r(0) = r_0 > 0$ is continuous for $t \in [0,\infty)$.*

Proof 3.2 *The local continuity of $r(t)$ is a consequence of general results on SDE with local Lipschitz coefficients (see e.g. [17]). From (6) it follows that*

$$u_t = u_0 + \alpha t - \int_0^t \left(e^{2u_s} + \frac{1}{2}\eta^2 \right) ds + \eta B_t \leq u_0 + \alpha t + \eta B_t,$$

which implies that u_t and hence r_t does not explode, for any $t \geq 0$.

Proposition 3.3 *There exists a unique (time continuous) global solution of (5).*

Proof 3.4 *From Theorem. 3.4.5 14 there exists a local maximal solution of (5) for all t, which is (locally) unique. From lemma 3.1 it follows that every process solving (5) is continuous for all t. From this it finally follows that the local maximal solutions form a unique global solution of (5).*

Corollary 3.5 *The solution process of (5) is a homogeneous diffusion process.*

Proof 3.6 *From the local version of proposition 9.3.1, [1], $r(t)$ is locally a homogeneous diffusion process. From the existence and uniqueness of a global solution it follows that this solution is again a homogeneous diffusion process.*

Lemma 3.7 *For $\eta > 0$, there exists a stationary probability density $p(r)$ for the process $r(t)$ given by (5). It has the form*

$$p(r) = N^{-1} r^{\frac{2(\alpha - \eta^2)}{\eta^2}} e^{-\frac{r^2}{\eta^2}} \tag{7}$$

where $N = \int_0^\infty r^{\frac{2(\alpha-\eta^2)}{\eta^2}} e^{-\frac{r^2}{\eta^2}} dr$.

Proof 3.8 *The existence of a stationary (i.e. invariant) measure (not necessarily finite) for (6) is already guaranteed by the homogeneity of the (one dimensional) equation. The Fokker-Planck equation for (6) has the form*

$$\frac{\partial p}{\partial t} + \frac{\partial}{\partial r}\left((\alpha r - r^3) p \right) - \frac{\eta^2}{2} \frac{\partial^2}{\partial r^2}(r^2 p) = 0$$

One sees easily that p as given by (7) is a stationary solution of the latter equation.

We point out that in the low noise limit $\eta \to 0$ the density $p(r)$ is distributed around its maximum near the unperturbed value $r = \sqrt{\alpha}$.
At this point we state that, as it was expected, random perturbations of a constant input current in (2) translate into irregular firing of the model neuron if $\gamma \neq 0$. If $\Omega \equiv \dot{\varphi}$ denotes the firing rate of the neuron then one obtains the following rate distribution:

$$p(\Omega) = N \left(\frac{\Omega - \omega}{\gamma} \right)^{\frac{\alpha - \eta^2}{\eta^2}} e^{-\left(\frac{\Omega - \omega}{\gamma \eta^2} \right)} \tag{8}$$

For the following analysis we will always assume that $\eta \ll \alpha$.

Remark 3.9 *(Stratonovich's interpretation). Considering a white noise perturbation of the parameter α physicists often write*

$$\dot{r} = (\alpha + \eta \xi_t) r - r^3, \tag{9}$$

where ξ_t denotes the Gaussian white noise.
A physical interpretation of (9) *leads to the differential equation*

$$dr = (\alpha - r^2) r \, dt + \eta r \circ dB_t$$

and the corresponding Itô differential equation of the form

$$dr = (\alpha + \frac{1}{2}\eta^2 - r^2) r \, dt + \eta r \, dB_t. \tag{10}$$

Equation (10) *is equivalent to* (5a) *with the transformation $\alpha' = \alpha + \frac{1}{2}\eta^2$. Thus, in the small noise limit, both interpretations are up to $\frac{1}{2}\eta^2$ equivalent.*
□

4 Weakly connected network of oscillators

In the case of a weekly connected network of N oscillators we correspondingly study a stochastic generalisation of equation (3) with the choice of parameters: $\alpha_k = \alpha, \omega_k = \omega > 0 \forall k, 0 < \eta \ll \alpha$ with α time independent.

$$dz_i = (\alpha + i\omega) z_i dt - z_i |z_i|^2 dt + \sum_j c_{ij} z_j dt + i \eta r_i dB_t^i, \tag{11}$$

wherein $c_{ij} = |c_{ij}| e^{i\psi_{ij}},\ \psi_{ij} \in [0, 2\pi),\ z_i \in \mathbb{C},\ i, j = 1..N.$.
Written in polar coordinates (r_i, ϕ_i) (so that $z_i = r_i e^{i\phi_i}$) one obtains from (11)

$$dr_i = \alpha r_i dt - r_i^3 dt + \eta r_i dB_t + \sum_{k \neq i} |c_{ik}| r_k \cos(\phi_k + \psi_{ik} - \phi_i) dt \tag{12}$$

$$d\phi_i = \omega dt + \frac{1}{r_i}\sum_{k\neq i}|c_{ik}|r_k \sin(\phi_k + \psi_{ik} - \phi_i)dt, \qquad i = 1..N \tag{13}$$

In the following we assume that the interconnections between the oscillators are weak in the sense that $\sum |c_{kl}|r_k \ll \sqrt{\alpha}$ where $r = \sqrt{\alpha}$ is the stationary solution of equation (2).

4.1 Independent radial dynamics

In this section we consider the case where the connections between neurons are of the particular form

$$c_{ij} = \mathrm{i}|c_{ij}|\sin\psi_{ij} \in \mathrm{i}\mathbb{R} \tag{14}$$

(for the analysis of the general case see section 4.2). In this situation the radial components r_i are independent:

$$dr_i = \alpha r_i dt - r_i^3 dt + \eta r_i dB_t^i, \quad i = 1\dots N \tag{15}$$

For the phases ϕ_i one has (for $r_i > 0$)

$$d\phi_i = \omega dt + \frac{1}{r_i}\sum_{k\neq i}|c_{ik}|r_k \sin(\phi_k + \psi_{ik} - \phi_i)dt, \qquad i = 1\dots N \tag{16}$$

Considering a possible synchronization of such a system it is convenient to define the phase difference of two oscillators

$$\chi_{ij} := \phi_i - \phi_j.$$

With these notations we have, for $r_i > 0 \ \forall\, i$,

$$d\chi_{ij} = \frac{1}{r_i}\sum_{k\neq i}|c_{ik}|r_k \sin(\chi_{ki} + \psi_{ik})dt - \frac{1}{r_j}\sum_{k\neq j}|c_{jk}|r_k \sin(\chi_{kj} + \psi_{jk})dt$$

$$= -\frac{1}{r_i}\sum_{k\neq i}|c_{ik}|r_k \sin(\chi_{ik} - \psi_{ik})dt - \frac{1}{r_j}\sum_{k\neq j}|c_{jk}|r_k \sin(\chi_{kj} + \psi_{jk})dt$$

$$= -\left(\frac{1}{r_i}\sum_{k\neq i}|c_{ik}|r_k \sin(\chi_{ik} - \psi_{ik}) + \frac{1}{r_j}\sum_{k\neq j}|c_{jk}|r_k \sin(\chi_{kj} + \psi_{jk})\right) dt$$

Remark 4.1 *From the stationary distribution* (7) *it is possible to compute the mean* $\mathbb{E}\left[\frac{r_k}{r_l}\right] =: \mu_0$. *One finds that* $\mu_0 = \lambda\frac{\Gamma^2(\lambda)}{\Gamma^2(\lambda+\frac{1}{2})}$, *where* Γ *is the Gamma function and* $\lambda \equiv \frac{\alpha-\eta^2}{\eta^2}$, *and further that* $\mu_0 \underset{\lambda\to\infty}{\to} 1$ *by using Stirling's formula.*

Assuming η to be small we approximate $\frac{r_k}{r_i}$ by a process fluctuating like a Gaussian white noise $\xi_t^{ki}(t) = $ "$\frac{dB_t^{ki}}{dt}$" of constant intensity $\upsilon > 0$ around μ_0. We write then

$$\frac{r_k}{r_i}(t) = \mu_0 + \upsilon \xi_t^{ki} \tag{17}$$

where the ξ_t^{ki} are all stocastically independent.
From this one has

$$d\chi_{ij} = -\mu_0 \left(\sum_{k\neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k\neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) dt \tag{18}$$

$$-\upsilon \sum_{k\neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) dB_t^{ki} - \upsilon \sum_{k\neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) dB_t^{kj}$$

with the initial condition $\chi_{ij}(0) = \Phi_{ij}$.
For the analysis of synchronizability the existence of the solution $\chi_{ij} = 0$ $\forall j, i < j$ of (18) is of particular interest.

Remark 4.2 *The solution of equation (18) with initial condition $\chi_{ij}(0) = \Phi_{ij}$ is unique in $[0, \infty)$ (on the basis of the known results on SDE with bounded coefficients, see, e.g. [17]).*

Remark 4.3 $\chi \equiv 0$ *is a solution of equation (18) if and only if $\psi_{ij}, \psi_{ji} \in \{0, \pi\}$ for all $i < j$ (this statement follows immediately setting $\chi_{ij} = 0$ in (18) and using the independence of the dB_t^{ij}).*

The corresponding Fokker-Planck equation for the probability density of the phase differences $\chi \equiv (\chi_{12}, \chi_{13}, \ldots, \chi_{N-1,N})$ in (18) is given by

$$-\mu_0 \sum_{i<j} \frac{\partial}{\partial \chi_{ij}} \left[\left(\sum_{k\neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k\neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) p \right] -$$

$$-\frac{\upsilon^2}{2} \sum_{i<j} \sum_{m<n} \frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} \left[\left(\sum_{k\neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k\neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) \times \right.$$

$$\left. \times \left(\sum_{l\neq m} |c_{ml}| \sin(\chi_{ml} - \psi_{ml}) + \sum_{l\neq n} |c_{nl}| \sin(\phi_{ln} + \psi_{nl}) \right) p \right] = \frac{\partial}{\partial t} p. \tag{19}$$

Definition 4.4 *Consider a network of* N *neurons. Let the dynamics of the neurons be given by (15) and (16). Then the matrix* $C := (c_{ij})_{1\leq i,j\leq N}$ *is called the network matrix.*

Recall the classical notion of stochastic stability (cf. [9]):

Definition 4.5 *(Stochastic stability) The solution identically zero of the equation*

$$dX(t) = b(X,t)dt + \sigma(t,X)dB_t \tag{20}$$

is said to be weakly stable in probability (for $t \geq t_0$*) if for every* $\epsilon > 0$, $\delta > 0$ *there exists an* $r > 0$ *such that if* $t > t_0$ *and* $|x_0| < r$*, then*

$$P\{|X(t,t_0,x_0)| > \epsilon\} < \delta. \tag{21}$$

It is said to be (weakly) asymptotically stable if

$$\lim_{t\to\infty} P\{|X(t,t_0,x_0)| > \epsilon\} = 0. \tag{22}$$

(where $X(t,t_0,x_0)$ *is the solution of* (20) *for* $t \geq t_0$ *with initial condition* $X(t_0,t_0,x_0) = x_0$*). It is (weakly) stable in the large if for all* x_0 *there is a* $t_0(x_0,\epsilon,\delta)$ *such that for all* $t > t_0$ *equation (21) holds.*

Using definition 4.5 we are able to formulate a definition of synchronizability of the considered stochastic oscillator network.

Definition 4.6 *Consider a network of* N *oscillators with a given network matrix* C*. Let the dynamics of the phase differences be given by (18). Then we say that the network is synchronizable if a stationary solution of equation (19) exists and for almost all initial conditions* $\chi(0) = (\Phi_{12},\ldots,\Phi_{(N-1)N})$ *it has the form* $p(\chi) = \prod_{i<j} \delta(\chi_{ij})$ *(with* δ *the Dirac distribution).*

Remark 4.7 *Definition 4.5 is equivalent to the (weak) asymptotic stability in the large of the trivial solution* $\chi \equiv 0$*.* □

In other words, a system of oscillators organized in the way given by the network matrix C is able to reach a synchronized state, provided that the oscillators are activated in a proper way, i.e. the assumptions in equations (11) to (18) are satisfied. It means in particular that if e.g. the α_k are time dependent, then it may happen that $\alpha_k(t > t_0) < 0$ for some k and t in a way that the system will not synchronize.
A first result is given by the following proposition giving a condition for the network not to be strongly synchronizable.

Proposition 4.8 *Consider a network of N oscillators with a given network matrix C. Let the dynamics of the phase differences be given by (18). Define*

$$A(\chi) := -\mu_0 \sum_{i<j} \left(|c_{ij}| \cos(\chi_{ij} - \psi_{ij}) + |c_{ji}| \cos(\chi_{ij} + \psi_{ji}) \right)$$

Assume that there is a pair i, j such that $|c_{ij}| > 0$ (note that $A(0)$ is a real constant depending on the network matrix C only). Then one has the following implication:

$A(0) > 0 \Rightarrow$ *the network is not synchronizable.*

Proof 4.9 *Consider a differential operator*

$$\mathcal{L} = -\sum_{ij} a_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i} + c(x), \qquad a_{ij} = a_{ji}$$

with smooth coefficients, in a bounded open subset U of $\mathbb{R}^N$ and zero boundary conditions. It is known (see e.g. [5], Thm. 6.5.2. for the principal eigenvalue for nonsymmetric elliptic operators) that if $c \geq 0$ in $U \subset \mathbb{R}^N$ then one has for all eigenvalues λ of $\mathcal{L}$ that $Re(\lambda) > 0$. Thus, the only solution p of $\mathcal{L}p = 0$ with zero boundary conditions on ∂U is the trivial one, identical to zero. But, on the other hand the definition 4.6 of synchronizability is equivalent to the existence of a nontrivial solution $p(\chi)$ on every open set U such that $0 \in U$. From this, synchronizability cannot occur if $c \geq 0$.
Consider the Fokker-Planck equation (19). After a corresponding transformation one finds that

$$c(\chi) = -\mu_0 \sum_{i<j} \frac{\partial}{\partial \chi_{ij}} \left(\sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) - \tag{23}$$

$$-\frac{v^2}{2} \sum_{i<j} \sum_{m<n} \frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} \left(\sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) \times$$

$$\times \left(\sum_{l \neq m} |c_{ml}| \sin(\chi_{ml} - \psi_{ml}) + \sum_{l \neq n} |c_{nl}| \sin(\chi_{ln} + \psi_{nl}) \right) =: A(\chi) + v^2 \epsilon_1(C)$$

with $\epsilon_1(C) \in \mathbb{R}$ *and*

$$\begin{aligned} A &:= -\mu_0 \sum_{i<j} \frac{\partial}{\partial \chi_{ij}} \left(\sum_{k=i+1}^{j} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k=i}^{j-1} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) \\ &= -\mu_0 \sum_{i<j} \left(|c_{ij}| \cos(\chi_{ij} - \psi_{ij}) + |c_{ji}| \cos(\chi_{ij} + \psi_{ji}) \right) \end{aligned}$$

From the assumptions of small noise one has that $\mu_0 \gg \upsilon$. *With this*

$$c(0) = A(0) + \upsilon^2 \epsilon_1(c, 0) \approx A(0)$$

in the sense that $c(0) > 0$ *if* $A(0) > 0$ *and if the noise is chosen small enough.*
Therefore, for an appropriately small υ *one has that* $c(0) \approx A(0)$. *It suffices thus to consider* $A(0)$ *instead of* $c(0)$. *Let* $A(0) > 0$. *From the smoothness of* $A(\chi)$ *we can deduce the existence of an open set* U, $0 \in U$ *such that* $c(\chi) \geq 0$ *and thus there is no nontrivial solution of (19) in* U *with zero boundary conditions. This gives the asserted implication.*

4.1.1 Example

Consider a network as in the assumptions of proposition 4.8. Let $\psi_{ij} = \pi$ for all $i < j$ (only inhibitory couplings). Then one has

$$A(0) = -\mu_0 \sum_{i<j} \left(|c_{ij}| \cos(\psi_{ij}) + |c_{ji}| \cos(\psi_{ji}) \right) = \mu_0 \sum_{i,j} |c_{ij}| > 0.$$

The network is thus not synchronizable independently of the choice of the linking strengths $|c_{ij}|$. □

Still we are looking for conditions which guarantee synchronizability.

Notation 4.10 *Let the drift coefficient in (19) be denoted by* $\mu_0 D_1$, *i.e.*

$$(D_1)_{ij} \equiv \sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} |c_{ik}| \sin(\chi_{kj} + \psi_{jk}) \qquad (24)$$

and let the diffusion coefficient in (19) be denoted by $\frac{\upsilon^2}{2} D_2$. *One has that* $D_2 = D_1 D_1^T$.

Lemma 4.11 *$p(\chi) = \delta(0) \equiv \prod_{i<j} \delta(\chi_{ij})$ is a (weak) solution of the stationary case of (19), i.e.*

$$
\begin{aligned}
&-\mu_0 \sum_{i<j} \frac{\partial}{\partial \chi_{ij}} \left[\left(\sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) p \right] - \\
&- \frac{\upsilon^2}{2} \sum_{i<j} \sum_{m<n} \frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} \left[\left(\sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) \times \right. \\
&\times \left. \left(\sum_{l \neq m} |c_{ml}| \sin(\chi_{ml} - \psi_{ml}) + \sum_{l \neq n} |c_{nl}| \sin(\phi_{ln} + \psi_{nl}) \right) p \right] = 0,
\end{aligned}
\tag{25}
$$

if and only if $D_1(0) = 0$.

Proof 4.12 *The statement is equivalent to the statement that for all test functions $f \in C_0^\infty(\mathbb{R}^{N(N-1)/2})$ the integrals of the left hand side of* (25) *with respect to f is zero. But the integral on the l.h.s. is equal to*

$$
\begin{aligned}
&- \int \left(\mu_0 \sum_{i<j} \frac{\partial}{\partial \chi_{ij}} [(D_1)_{ij} p] + \frac{\upsilon^2}{2} \sum_{i<j} \sum_{m<n} \frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} [(D_2)_{ij\ mn} p] \right) f \, d\chi = \\
&= -\mu_0 \sum_{i<j} \int \frac{\partial}{\partial \chi_{ij}} [(D_1)_{ij} p(\chi)] f \, d\chi - \frac{\upsilon^2}{2} \sum_{i<j} \sum_{m<n} \int \frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} [(D_2)_{ij\ mn} p(\chi)] f \, d\chi = \\
&= -\mu_0 \sum_{i<j} \int (D_1)_{ij} \, p(\chi) \left(\frac{\partial}{\partial \chi_{ij}} f \right) (\chi) \, d\chi - \\
&\quad - \frac{\upsilon^2}{2} \sum_{\substack{i<j \\ m<n}} \int (D_2)_{ij\ mn} \, p(\chi) \left(\frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} f \right) (\chi) \, d\chi.
\end{aligned}
\tag{26}
$$

We set $p = \delta(0)$ and obtain for the last expression of (26)

$$
\begin{aligned}
&- \mu_0 \sum_{i<j} (D_1)_{ij}(0) \left(\frac{\partial}{\partial \chi_{ij}} f \right) (0) - \\
&- \frac{\upsilon^2}{2} \sum_{\substack{i<j \\ m<n}} (D_1)_{ij}(0) (D_1)_{mn}(0) \left(\frac{\partial^2}{\partial \chi_{ij} \partial \chi_{mn}} f \right) (0)
\end{aligned}
\tag{27}
$$

This is clearly zero for all $f \in \mathcal{C}_0^\infty(\mathbb{R}^{N(N-1)/2})$ *if* $D_1(0) = 0$*. Vice versa, if* (27) *vanishes for all* $f \in C_0^\infty(\mathbb{R}^{N(N-1)/2})$*, for all* $i < j$ *one can choose an* f *such that* $\frac{\partial^2}{\partial\chi_{kl}\partial\chi_{mn}} f|_{\chi=0} = 0$ *for all* k, l, m, n *and* $\frac{\partial}{\partial\chi_{kl}} f|_{\chi=0} = 0$ *for all* $k \neq i$*,* $l \neq j$ *and* $\frac{\partial}{\partial\chi_{ij}} f|_{\chi=0} \neq 0$*. From this, we see that then* $D_1(0) = 0$ *for all* $i < j$*.*

Corollary 4.13 *In particular, if* $\psi_{ij} \in \{0, \pi\}$ *for all* i, j *then* $p = \delta(0)$ *is a solution of (19).*

Proof 4.14 *This follows immediately from the definition of* D_1 *(equation (24)).*

According to the definition 4.6 a system (given by the set of parameters c_{ij} and ψ_{ij}) is synchronizable if for the asymptotic distribution p one has $p(\chi) = \delta(0)$ for all starting configurations $\chi(0) = (\Phi_{12}, \ldots, \Phi_{(N-1)N})$ of the phases. In a more general setting one has the following lemma.

Lemma 4.15 *Consider the system of SDE*

$$dx(t) = \mu f(x)dt + \nu \sum_{i=1}^{r} g_i(x)dB_t^i, \quad x(\cdot) \in \mathbb{R}^d, \quad x_i(0) > 0 \tag{28}$$

for all $i = 1, \ldots, N$*, some* $r \in \mathbb{N}$*, with assumptions on continuity and differentiability of* f *and* g *such that a solution exists and where* $f(0) = g_i(0) = 0$ *for all* i *and* $\mu \gg \nu > 0$ *(*μ*,* ν *being parameters). The solution* $x \equiv 0$ *of* (28) *is asymptotically stable in probability if the linearized system*

$$dx(t) = \mu\, df(0)x\, dt + \nu \sum_{i=1}^{r} dg_i(0)x\, dB_t^i, \quad x_i(0) > 0, i = 1, \ldots, d \tag{29}$$

is stable in the large and df*,* dg *are bounded and continuous at* $x = 0$*.*

Proof 4.16 *See e.g. 9.*

Lemma 4.17 *Assuming in addition to the assumptions in lemma 4.15 that* g *in* (29) *is bounded and* dg *is continuous at* $x = 0$ *we have that the system given by equation* (29) *is stable in the large if the corresponding deterministic system*

$$\dot{y} = df(0)y \tag{30}$$

has an asymptotically stable fixed point at $y = 0$*.*

Proof 4.18 *Let* $\lim_{t\to\infty} y(t) = 0$ *for all* $y(0)$ *such that* $y_i(0) > 0$, $i = 1, \ldots, N$, *for a solution of* (30). *For a process* $x(t)$ *solving equation* (29) *one has that*

$$\frac{d}{dt}\mathbb{E}[x(t)] \equiv \dot{m}_t = \mu\, df(0) m_t. \tag{31}$$

From this it follows from the assumptions on f *that* $\lim_{t\to\infty} m_t = 0$ *for all* $x(0) > 0$.
For the second moment $Q(t) \equiv \mathbb{E}[x(t)x^t(t)]$ *one has ([8], pp 310-316)*

$$\frac{d}{dt}Q = \mu\, df(0)Q + \mu\, Qdf(0)^t + O(\nu^2). \tag{32}$$

From this, with $\nu \ll \mu$, *and the assumptions on* $y = 0$ *being an asymptotically stable solution fixed point of* (30) *we have also* $\lim_{t\to\infty} Q(t) = 0$ *for all* $x(0) > 0$ *and thus one obtains the stability in the large of system* (29).

We finally obtain the following result.

Proposition 4.19 *Consider the system of oscillators whose phase dynamics is given by equations (18). Let a mapping* $F(\chi) : \mathbb{R}^{N(N-1)/2} \to \mathbb{R}^{N(N-1)/2}$ *be given by*

$$(F(\chi))_{ij} := -\mu_0 \left(\sum_{k\neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k\neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) \equiv -\mu_0 (D_1)_{ij}.$$

Note that (ij) *is treated as a single index counting the pairs with* $i, j = 1 \ldots N$, $i < j$.
Moreover let $\psi_{ij} \in \{0, \pi\}$ *for all* i, j. *Then the system is synchronizable in the sense of definition 4.6 if*

1. $(D_1)(0) = 0$,
2. $x = 0$ *is an asymptotically stable state of the system* $\frac{dx}{dt} = F(x)$, $x \in \mathbb{R}^{N(N-1)/2}$.

Proof 4.20 *From lemma 4.11 it is known that the synchronized solution of (18) exists iff* $D_1(0) = 0$. *From the assumptions on* ψ_{ij} *one knows that the only absorbing states of the system are solutions of the form* $\chi_{ij} \in \{0, \pi\}$ *for all* $i < j$. *From the analogy with a system of two oscillators (see example 4.1.2 below) and from the fact that* $F(0) = -F(\pi)$ *(setting* $F(\pi) \equiv F(\pi, \ldots, \pi)$*) one knows moreover that if* $\chi \equiv 0$ *is stable then any* *other solution with $i < j$ such that $\chi_{ij} \equiv \pi$ is not stable*[1]*. Therefore if $\chi \equiv 0$ is stable then it is the unique stable solution. On the other hand, by lemma 4.15 and 4.17, this solution is asymptotically stable if the same holds for the corresponding deterministic system $\dot{x} = F(x)$.*

A converse statement is given in the following lemma.

Lemma 4.21 *Let $\bar{\nu} > 0$ exist, such that $x \equiv 0$ of (28) is asymptotically stable for all $0 < \nu \leq \bar{\nu}$. Then the solution $x = 0$ of the corresponding deterministic system $\dot{x} = f(x)$ is stable.*

Proof 4.22 *Let $\bar{\nu}$ be as is the assumptions. Assume $\dot{x} = f(x)$ is asymptotically unstable at $x = 0$. Then from lemma 4.17 we know that we can choose a $\nu < \bar{\nu}$ such that (29) and thus (28) as well are asymptotically unstable. But this is a contradiction to the assumptions.*

From lemma 4.21 we conclude that in our setting a network of deterministic oscillators without a stable synchronized state cannot be stabilized by an additional noise term.

4.1.2 Example

Consider a system consisting of two neurons. Let $\psi_{ij} \in \{0, \pi\}$, $i, j = 1 \ldots N$. Then (18) consists of only one equation and one has the following:

1. $D_1(0) = |c_{12}| \sin \psi_{12} + |c_{21}| \sin \psi_{21} = 0$,
2. $F'(0) = -\mu_0 |c_{12}| \cos \psi_{12} - \mu_0 |c_{21}| \cos \psi_{21} \overset{!}{<} 0$.

Thus one has the following cases:

1. Excitatory links ($\psi_{12} = \psi_{21} = 0$) lead to synchronizability;
2. Inhibitory links ($\psi_{12} = \psi_{21} = \pi$) hinder synchronization;
3. The behaviour of the system with mixed (excitatory and inhibitory) links depends on their strengths $|c_{ij}|$.

□

[1]Considering a system of N neurons assume that a solution $\chi \neq 0$ is stable. Then the neurons can be grouped in such way that one obtains two synchronized populations of N_1 and N_2 neurons, $N_1 + N_2 = N$. The phase difference between the groups can be considered as the only free parameter of the system. Thus, one obtains a system analogous to the one of two weakly connected oscillators with a stable non synchronous solution. As it is shown in example 4.1.2 below, the synchronous solution of such a system is not stable.

4.2 Interdependent radial dynamics

Consider the case $\epsilon = 1$ in (14), i.e. the system given by the equations

$$dr_i = \alpha r_i dt - r_i^3 dt + \eta r_i dB_t + \sum_{k \neq i} |c_{ik}| r_k \cos(\phi_k + \psi_{ik} - \phi_i) dt \tag{33}$$

and

$$d\phi_i = \omega dt + \sum_{k \neq i} |c_{ik}| \frac{r_k}{r_i} \sin(\phi_k + \psi_{ik} - \phi_i) dt, \qquad i = 1..N \tag{34}$$

Consider equation (33). Assuming $\sum |c_{ik}| r_k \ll \sqrt{\alpha}$ we would like to approximate $\frac{r_k}{r_i}$, similarily to the case of independent radial dynamics, as a fluctuation of white noise around a mean μ_{ki}. In order to estimate μ_{ki} we consider a stationary point of (33) at $\phi_k - \phi_i = 0 \quad \forall k, i$, because the state of zero phase differences is of particular interest for our analysis of synchronizability. If the stationary point exists, then it is given by the solution of the nonlinear equation

$$\alpha r + Cr - r^3 = 0, \tag{35}$$

where $r \equiv (r_1, \ldots, r_N), r^3 \equiv (r_1^3, \ldots, r_N^3)$.
The solution of (35) can be found using e.g. any appropriate numerical method up to the desired accuracy. We use the first step of the Newton method as a rough approximation:

$$\mu_{ki} \approx \frac{1 + \frac{1}{2\alpha} \sum_{i \neq k} |c_{ik}| \sin \psi_{ik}}{1 + \frac{1}{2\alpha} \sum_{j \neq k} |c_{jk}| \sin \psi_{jk}}. \tag{36}$$

With this approximation one obtains for the phase differences

$$\begin{aligned} d\chi_{ij} \quad = \quad & - \left(\sum_{k \neq i} \mu_{ik}(\chi) |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} \mu_{jk}(\chi) |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right) dt \\ & - \upsilon \sum_{k \neq i} |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) dB_t^{ki} - \upsilon \sum_{k \neq j} |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) dB_t^{kj} \end{aligned} \tag{37}$$

Let D_1 denote the drift coefficient in equation (37). Then one has that the choice $\psi_{ij} \in \{0, \pi\}$ for all i, j guarantees that $D_1(0) = 0$.
Defining $\left(\widetilde{F}(\chi)\right)_{ij} := - \left(\sum_{k \neq i} \mu_{ik}(\chi) |c_{ik}| \sin(\chi_{ik} - \psi_{ik}) + \sum_{k \neq j} \mu_{jk}(\chi) |c_{jk}| \sin(\chi_{kj} + \psi_{jk}) \right)$ one obtains the following corollary.

Corollary 4.23 *Consider the system given by (37). Let* $\psi_{ij} \in \{0, \pi\}$ *for all* i, j*. Then the system is synchronizable if* $\chi = 0$ *is an asymptotically stable solution of*

$$\dot{\chi} = \widetilde{F}(\chi) \tag{38}$$

Proof 4.24 *The assertion follows immediately from proposition 4.19 by replacing* $F(\chi)$ *by* $\widetilde{F}(\chi)$ *and from the fact that* $sgn(\mu_{ij}(0)) = sgn(\mu_{ij}(\pi))$ *and* $\widetilde{F}(0) = -\widetilde{F}(\pi)$*.*

Remark 4.25 *Let* $\psi_{ij} \in \{0, \pi\}$ *for all* i, j*. Then one has that*

$$\frac{\partial(\widetilde{F})_{ij}}{\partial \chi_{mn}}|_{\chi_{mn}=0} = -\left[\delta_{im}\mu_{mn}(0)|c_{mn}|\cos\psi_{mn} + \delta_{jn}\mu_{nm}(0)|c_{nm}|\cos\psi_{nm}\right]. \tag{39}$$

□

Consider the deterministic system given by equations (33) (with $\eta = 0$) and (38).
Recall that corollary 4.23 makes use of the approximation for the asymptotic radii μ_i. In the following we shall investigate the stability properties of the deterministic system ((33),(38)) without this approximation.
One has that its linearization gives the Jacobi matrix J, taken at $\chi = 0$ and r at the asymptotic equilibrium (if it exists), of the form

$$J = \left(\begin{array}{c|c} \left(\frac{\partial (G)_i}{\partial r_m}\right)_{im} & 0 \\ \hline 0 & \left(\frac{\partial (\widetilde{F})_{ij}}{\partial \chi_{mn}}\right)_{ijmn} \end{array} \right), \qquad i < j, m < n \tag{40}$$

where G is given by $(G)_i = \alpha r_i - r_i^3 + \sum\limits_{k \neq i} |c_{ik}| r_k \cos(\phi_k + \psi_{ik} - \phi_i)$, and where we assume that $\psi_{ij} \in \{0, \pi\}$ for all i, j.

Lemma 4.26 *The eigenvalues of* $\left(\frac{\partial (G)_i}{\partial r_m}\right)_{im}$ *coincide with its diagonal elements and the corresponding eigenvectors coincide with the unit vectors* e_i *up to* $O(\sum_{i \neq k} |c_{ik}|)$ *(in the sense of a formal expansion in powers of* $\sum_{i \neq k} |c_{ik}|$*).*

Proof 4.27 *One has*

$$\left(\frac{\partial (G)_i}{\partial r_j}\right)_{ij} = \delta_{ij}(\alpha - 3r_i^2) + (1-\delta_{ij})|c_{ij}|\cos\psi_{ij}.$$

With the decomposition $\left(\frac{\partial (G)_i}{\partial r_j}\right)_{ij} = (D)_{ij} + (N)_{ij}$ *where* $(D)_{ij} \equiv \delta_{ij}(\alpha - 3r_i^2)$ *and* $(N)_{ij} \equiv (1-\delta_{ij})|c_{ij}|\cos\psi_{ij}$ *it follows that*

$$\left(\frac{\partial (G)_i}{\partial r_j}\right) e_i = (\alpha - 3r_i^2) + O(\sum_{i\neq k}|c_{ik}|).$$

On the other hand let v *be such that* $\left(\frac{\partial (G)_i}{\partial r_j}\right) v = \gamma v$, $\|v\| = 1$, $\gamma \in \mathbb{R}$. *Then* $\left(\frac{\partial (G)_i}{\partial r_j}\right) v = \gamma v = (D+N)v = Dv + O(\sum_{i\neq k}|c_{ik}|)$. *This holds iff* $Dv = \gamma v + O(\sum_{i\neq k}|c_{ik}|)$ *and therefore* $v = e_i + O(\sum_{i\neq k}|c_{ik}|)$.

Remark 4.28 *From lemma 4.26 we can conclude that, for* $\max_{i,j}|c_{ik}|$ *chosen sufficiently small, one has that the eigenvalues* γ_i *of* $\left(\frac{\partial (G)_i}{\partial r_j}\right)$ *are all negative.*

Proof 4.29 *One has* $\gamma_i = \alpha - 3r_i^2 + O(\sum|c_{ik}|)$ *(we recall that* r_i *is the equilibrium radius of the i-th neuron). On the other hand one has from the assumptions on weak connectivity* $r_i = \sqrt{\alpha} + O(\sum|c_{ik}|)$. *From this it follows that* $\gamma_i = -2\alpha + O(\sum|c_{ik}|) < 0$ *for an appropriate choice of the linking strengths* $|c_{ij}|$.

From remark 4.28 one concludes that in the case considered so far, i.e. where $\sum|c_{ik}| \ll \sqrt{\alpha}$, in order to analyze the stability of the trivial solution $\chi \equiv 0$ it suffices to consider the submatrix given by equation (39).

4.3 Short time synchronization

Allowing the synaptic connections c_{ij} to be complex, i.e. $\psi_{ij} \neq 0, \pi$, one obtains systems being synchronizable in the deterministic case but not synchronizable in the sense of definition 4.6 if noise is added. This phenomenon shall be illustrated by the following example.

4.3.1 Example

Consider a system of two oscillators (given by equations (33) and (37)) with the following choice of parameters: $\alpha \gg |c_{12}| = |c_{21}| =: c$, $\psi_{12} = \psi_{21} = \frac{\pi}{4}$. This system has in the deterministic case ($\eta = 0$ in equation (33)) the stable fixed point at $\chi = 0$. The two oscillators are able to synchronize. Nevertheless, because $\psi_{12} = \psi_{21} \neq 0, \pi$, one has that if $\eta \neq 0$ the system is not synchronizable in the sense of definition 4.6. Instead, one observes the following:

1. The phase difference χ_t is given by

$$d\chi_t = -c(\sin(\chi_t - \frac{\pi}{4}) + \sin(\chi_t + \frac{\pi}{4})) - vc\left(\sin(\chi_t - \frac{\pi}{4})dB_t^1 + \sin(\chi_t + \frac{\pi}{4})dB_t^2\right)$$

2. For the mean value m_χ of the phase difference one has $\underset{t\to\infty}{lim}\; m_\chi = 0$. This follows from the stability of the deterministic fixed point $\chi = 0$.

3. Starting at a phase difference χ_0 close to zero one obtains the short time approximation

$$d\chi_t = vc \sin\frac{\pi}{4}(dB_t^1 - dB_t^2)$$

   and thus for short times t the process $\chi_t = a(B_t^1 - B_t^2)$ with $a \equiv vc\sin\frac{\pi}{4}$.

For B_t^i, $i = 1, 2$ we assume standard independent, one-dimensional Brownian motions with zero mean and variance t. With this one obtains the (time asymptotic) distribution density of the phase difference χ as centered around zero and with variance (approximately) equal to $2a^2t$. For short times t the distribution of phases is very narrow. In this context it is convenient to speak about possible short time synchronization of the oscillators and of weak synchronizability of the system.

5 Conclusions

Motivated by experimental observations of 'noisy' behaviour of neural activity (see e.g. [7, 22] and the references therein) we analyzed a stochastic generalization of a model of coupled oscillators. The theory of stochastic differential equation allows to define a synchronized stationary solution of stochastic oscillators in the sense of the (stochastic) stability of the $\chi \equiv 0$ solution, where χ denotes the phase difference between the oscillators. In this context we investigated the stability of synchronized states of a deterministic system of oscillators under small random perturbations. Using tools from the theory of SDE we were able to formulate general conditions for synchronizability of the noisy system. In particular for the low level noise limit we were able to derive necessary and sufficient conditions based on the synchronizability of the corresponding deterministic system. We have shown that for systems of (deterministic) oscillators where the natural phase differences ψ_{ij} ([10]) take specific values ($\psi_{ij} \in \{0, \pi\}$) the asymptotic synchronized states are stable with respect to small random perturbations of the bifurcation parameter (or 'input current').

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